POSITIVE-ENTROPY GEODESIC FLOWS ON NILMANIFOLDS

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Abstract. Let T_n be the nilpotent group of real $n \times n$ upper-triangular matrices with 1s on the diagonal. The hamiltonian flow of a left-invariant hamiltonian on T^*T_n naturally reduces to the Euler flow on $\mathfrak{t}_n^*$, the dual of $\mathfrak{t}_n = \mathrm{Lie}(T_n)$. This paper shows that the Euler flows of the standard riemannian and sub-riemannian structures of T_4 have transverse homoclinic points on all regular coadjoint orbits. As a corollary, left-invariant riemannian metrics with positive topological entropy are constructed on all quotients $D\backslash T_n$ where D is a discrete subgroup of T_n and $n \geq 4$.

1. Introduction

Let Σ be a nilmanifold, *i.e.* homogeneous space of a connected nilpotent Lie group G. Each homogeneous riemannian metric on G induces a locally-homogeneous metric on Σ. These riemannian geometries, which will be called *left-invariant*, are of interest in both geometry and dynamics. A basis question is

Question A: *Which left-invariant geodesic flows on a compact nilmanifold have zero topological entropy?*

A mistaken answer to question A appears in Theorem 3 of [9]. In [2], the first author showed that on 2-step nilmanifolds, all left-invariant geodesic flows have zero entropy. In [3], metrics on compact quotients of the 3-step nilpotent Lie group $T_4 \oplus T_3$ are constructed whose geodesic flows have positive topological entropy. The paper also speculated that the standard geodesic flow on T_4 also had such horseshoes. Montgomery, Shapiro and Stolin [10] investigated the standard *sub-riemannian* geodesic flow on T_4 [2]; they showed that it reduces to the Yang-Mills hamiltonian flow which is known to be algebraically non-integrable [14, 15].

Let us state the first result of the present paper. The Lie algebra of T_4, $\mathfrak{t}_4$, has the standard basis consisting of those 4×4 matrices X_{ij} with a unit in the i-th row and j-th column, $i < j$, and zeros everywhere else. A quadratic hamiltonian $h : \mathfrak{t}_4^* \to \mathbf{R}$ is *diagonal* if it is expressed as $h(p) = \sum_{i<j} a_{ij} \langle p, X_{ij} \rangle^2$ for some constants a_{ij}. The standard riemannian metric has $a_{ij} = 1$ for all i, j; the standard Carnot (subriemannian) metric has $a_{12} = a_{23} = a_{34} = 1$ and all other coefficients zero.

Theorem 1.1. *If $h : \mathfrak{t}_4^* \to \mathbf{R}$ is a diagonal hamiltonian with $a_{12}a_{13}a_{23}a_{34} \neq 0$ and $a_{13}a_{34} = a_{12}a_{24}$, then for all but at most countably many regular coadjoint orbits in $\mathfrak{t}_4^*$, the Euler vector field of h has a horseshoe. In particular, the Euler vector field of the standard riemannian metric (resp. sub-riemannian metric with $a_{13} \neq 0$) is analytically non-integrable.*

The condition that $a_{13}a_{34} = a_{12}a_{24}$ is only a device to simplify the proof: all nearby hamiltonians also have a horseshoe. We also show, by means of a numerical computation of an integral (see Table 1 in section 4), that when $a_{13} = 0$, the

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conclusions of Theorem 1.1 hold. This shows that the subriemannian geodesic flow of Montgomery, Shapiro and Stolin is real-analytically non-integrable. Related numerical computations (figure 1 in section 4) suggest that the hamiltonian h has a horseshoe on *every* regular coadjoint orbit provided only that $a_{12}a_{23}a_{34} \neq 0$ and $a_{13}a_{34} = a_{12}a_{24}$. Let us formulate a corollary to Theorem 1.1: Let $D < T_n$ be a discrete subgroup of T_n, $\Sigma = D \backslash T_n$ and $S\Sigma$ is the unit sphere bundle.

Theorem 1.2. *If $n \geq 4$, then there is a left-invariant geodesic flow $\phi_t : S\Sigma \to S\Sigma$ such that $h_{top}(\phi_1) > 0$.*

It appears likely that *all* left-invariant geodesic flows on $S\Sigma$ have positive topological entropy and are non-integrable with smooth integrals.

Theorem 1.2 is interesting from a riemannian point of view. Let (M, g) be a smooth (C^∞) riemannian manifold, and $\phi_t : SM \to SM$ the geodesic flow of g. For each $T > 0$ and $p, q \in M$, let $n_T(p, q)$ denote the number of distinct geodesics of length no more than T that join p to q. Mañé [8] showed that if M is compact then $h_{top}(\phi_1|SM) = \lim_{T\to\infty} T^{-1} \log \int_{M\times M} \, n_T(p, q)\, dpdq$. Thus, for the geodesic flows constructed here, for generic points p and q on a compact quotient of T_n, $n_T(p, q)$ grows exponentially fast. In constrast, Karidi showed that the volume growth on the universal cover of these manifolds is polynomial of degree $\frac{1}{6}n(n^2 - 1)$ [6].

Theorem 1.1 is proved by reducing the hamiltonian flow of ϕ_t on $T^*\Sigma$ to a hamiltonian flow on the coadjoint orbits of $\mathrm{Lie}(T_4)^*$. In the appropriate coordinate system, the reduced hamiltonian is a small perturbation of a hamiltonian on $\mathbf{R}^4$ that is the sum of an unforced Duffing hamiltonian and a forced linear system whose solutions can be expressed in terms of Legendre functions. The Poincaré-Melnikov technique developed in [7, 11] for autonomous Hamiltonian systems is adapted here to show that a suspended Smale horseshoe appears in the perturbed hamiltonian flow.

2. The construction on Lie$(T_4)^*$

In this section, we will first recall a number of key facts about geodesic flows and left-invariant hamiltonian systems on the cotangent bundle of a Lie group; for more details, see [5]. We will then reduce the equations of motion of a left-invariant geodesic flow on T^*T_4 to the equations of motion of a hamiltonian system on $T^*\mathbf{R}^2$.

2.1. **Poisson geometry of left-invariant hamiltonians.** A *Poisson manifold* is a smooth manifold M such that $C^\infty(M)$ is equipped with a skew-symmetric bracket $\{,\}$ that makes $(C^\infty(M), \{,\})$ into a Lie algebra of derivations of $C^\infty(M)$. The centre of $(C^\infty(M), \{,\})$ is traditionally called the set of *Casimirs*. If f is a Casimir then $X_f \equiv 0$ and f is a first integral of all hamiltonian vector fields. If the set of Casimirs of $(C^\infty(M), \{,\})$ are the constant functions, then we say that $(C^\infty(M), \{,\})$ is a *symplectic* manifold. In this case, the Poisson bracket naturally induces a closed, non-degenerate skew 2-form on M which is called a symplectic structure. We will say that a smooth map $f : M \to N$ is a *Poisson map* if $f^* : (C^\infty(N), \{,\}_N) \to (C^\infty(M), \{,\}_M)$ is a Lie algebra homomorphism.

The most basic example of a Poisson manifold that is also symplectic is provided by $T^*\mathbf{R} = \{(a, A) \ : a, A \in \mathbf{R}\}$ equipped with the Poisson bracket satisfying $\{a, A\}_{T^*\mathbf{R}} = 1$.

The dual space of a Lie algebra gives an example of a Poisson manifold that is not (in general) a symplectic manifold. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. $T_p^*\mathfrak{g}^*$ is identified with $\mathfrak{g}$ for all

$p \in \mathfrak{g}^*$. The Poisson bracket on $\mathfrak{g}^*$ is defined for all $f, h \in C^\infty(\mathfrak{g}^*)$ and $p \in \mathfrak{g}^*$ by

$$\{f, h\}(p) := -\langle p, [df_p, dh_p]\rangle, \tag{1}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbf{R}$ is the natural pairing. Recall that for $\xi \in \mathfrak{g}$, $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the linear map defined by $\langle \mathrm{ad}^*_\xi p, \eta\rangle = -\langle p, [\xi, \eta]\rangle$. $\mathrm{ad}^* : \mathfrak{g} \to gl(\mathfrak{g}^*)$ is the representation contragredient to the adjoint representation. For any $h \in C^\infty(\mathfrak{g}^*)$, the hamiltonian vector field $E_h = \{\cdot, h\}$ equals $-\mathrm{ad}^*_{dh_p} p$. The standard example of a hamiltonian vector field is obtained from a positive-definite linear map $\phi : \mathfrak{g}^* \to \mathfrak{g}$ by setting $h(p) = \frac{1}{2}\langle p, \phi(p)\rangle$, in which case $E_h(p) = -\mathrm{ad}^*_{\phi(p)} p$.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. The adjoint representation of G on $\mathfrak{g}$, $\mathrm{Ad}_g \xi = \frac{d}{dt}|_{t=0}\, g \exp(t\xi) g^{-1}$, induces the coadjoint representation $\langle \mathrm{Ad}^*_g p, \xi\rangle = \langle p, \mathrm{Ad}_{g^{-1}} \xi\rangle$ for all $p \in \mathfrak{g}^*$, $g \in G$ and $\xi \in \mathfrak{g}$. As each vector field $p \to \mathrm{ad}^*_\xi p$ is hamiltonian on $\mathfrak{g}^*$, with linear hamiltonian $h_\xi(p) = -\langle p, \xi\rangle$, the coadjoint action of G on $\mathfrak{g}^*$ preserves the Poisson bracket. The orbits of the coadjoint action are called the coadjoint orbits. Each coadjoint orbit is a homogeneous G-space, and *every* hamiltonian vector field on $\mathfrak{g}^*$ is tangent to each coadjoint orbit. For this reason, the Poisson bracket $\{,\}_{\mathfrak{g}^*}$ restricts to each coadjoint orbit, and is non-degenerate on each coadjoint orbit. Thus, the coadjoint orbits are naturally symplectic manifolds. A Casimir is necessarily constant on each coadjoint orbit, and in many cases (as in this paper) each coadjoint orbit is the common level set of all Casimirs.

The Poisson bracket on $\mathfrak{g}^*$ also arises in a natural way from the Poisson bracket on T^*G. The group G acts from the left on T^*G, and this action preserves the Poisson structure $\{,\}_{T^*G}$. The set of smooth left-invariant functions $C^\infty(T^*G)^G$ is therefore a Lie subalgebra of $C^\infty(T^*G)$ with respect to $\{,\}_{T^*G}$. This subalgebra is naturally identified with $(C^\infty(\mathfrak{g}^*), \{,\}_{\mathfrak{g}^*})$ as follows: the left-trivialization of $T^*G = G \times \mathfrak{g}^*$ induces the projection map $r : T^*G \to \mathfrak{g}^*$ onto the second factor; $r^*C^\infty(\mathfrak{g}^*) = C^\infty(T^*G)^G$ and r^* is a Lie algebra monomorphism.

The hamiltonian flow of a left-invariant hamiltonian H on T^*G therefore has the equations of motion:

$$X_H(g, p) = \left\{ \begin{array}{rcl} \dot{g} & = & T_e L_g dh(p), \\ \dot{p} & = & -\mathrm{ad}^*_{dh(p)}\, p, \end{array} \right. \tag{2}$$

Note that $\mathrm{d}r(X_H) = E_h$, where $h \in C^\infty(\mathfrak{g}^*)$ satisfies $r^*h = H$. The vector field E_h is called the *Euler vector field*. It is a natural reduction of X_H by G. If $h(p) = \frac{1}{2}\langle p, \phi(p)\rangle$ for a positive-definite linear map $\phi : \mathfrak{g}^* \to \mathfrak{g}$ then H is induced by a left-invariant metric on T^*G and X_H is the geodesic vector field.

Finally, if $D < G$ is a discrete subgroup, then $T^*\Sigma = \Sigma \times \mathfrak{g}^*$ where $\Sigma = D\backslash G$. The projection map $r : T^*G \to \mathfrak{g}^*$ is naturally left-invariant, so it factors through to a map $r_o : T^*\Sigma \to \mathfrak{g}^*$. If $H_o = r_o^* h$ for some $h \in C^\infty(\mathfrak{g}^*)$ then $Tr_o(X_{H_o}) = E_h$. Thus, the hamiltonian flow of a left-invariant hamiltonian on $T^*\Sigma$ always projects to a hamiltonian flow on $\mathfrak{g}^*$.

2.2. **Poisson geometry of T^*T_4.** Let $\mathfrak{t}_4$ denote the Lie algebra of T_4, so

$$\mathfrak{t}_4 = \left\{ \begin{bmatrix} 0 & x & z & w \\ 0 & 0 & y & u \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{bmatrix} \ :\ u, v, w, x, y, z \in \mathbf{R} \right\}.$$

For each $a \in \{u, v, w, x, y, z\}$, let $A \in \mathfrak{t}_4$ be the element obtained by setting a equal to one and all other coefficients equal to zero. Then $\{U, V, W, X, Y, Z\}$ is a basis of $\mathfrak{t}_4$ whose commutation relations given by: $[X, Y] = Z$,$[Y, V] = U$, $[X, U] = W$, $[Z, V] = W$, and all others are trivial or obtained by skew-symmetry.

Let $p_a : \mathfrak{g}^* \to \mathbf{R}$ be the linear function given by $p_a(p) = \langle p, A\rangle$ for $A \in \mathfrak{g}$ and all $p \in \mathfrak{g}^*$. From the definition of the Poisson bracket on $\mathfrak{t}_4^*$ (*c.f.* Eq. 1), along with the commutation relations, we conclude that:

$$\{p_x, p_y\} = -p_z,\ \{p_y, p_v\} = -p_u,\ \{p_x, p_u\} = -p_w,\ \{p_z, p_v\} = -p_w.$$

There are two independent Casimirs of $\mathfrak{t}_4^*$ are $K_1(p) = p_w$, $K_2(p) = p_w p_y - p_z p_u$. Let $K : \mathfrak{t}_4^* \to \mathbf{R}^2$ be defined by $K = (K_1, K_2)$. The level sets of K are the coadjoint orbits of T_4's action on $\mathfrak{t}_4^*$ and will be denoted by $\mathcal{O}_k$, where $k = (k_1, k_2)$. We will say that $\mathcal{O}_k$ is a *regular coadjoint orbit* if $k_1 k_2 \neq 0$.

Lemma 2.1. *Let $\mathcal{O}_k$ be a regular coadjoint orbit. Then $\mathcal{O}_k$ is symplectomorphic to $(T^*\mathbf{R})^2$ equipped with its canonical symplectic structure.*

Proof. The Poisson bracket on $\mathfrak{t}_4^*$ restricts to $\mathcal{O}_k$. We will denote the restricted bracket by $\{,\}_k$. Let $(T^*\mathbf{R})^2 = \{(a, A, b, B) \ :\ a, A, b, B \in \mathbf{R}\}$. The canonical Poisson bracket, which we denote by $[,]$, satisfies $[a, A] = 1$, $[b, B] = 1$ and all other brackets are zero.

Let λ and μ be two non-zero parameters (the parameters are included because we will further transform coordinate systems). The map $f_k(p) = (a, A, b, B)$ defined by

$$a = -\lambda p_x,\ A = (k_1\lambda)^{-1} p_u,\ b = -\mu p_v,\ B = (k_1\mu)^{-1} p_z$$

is a diffeomorphism of $\mathcal{O}_k$ onto $T^*\mathbf{R}^2$. Indeed, f_k is clearly smooth. And $g_k(a, A, b, B) = (p_u, \dots, p_z)$ defined by

$$p_v = -\mu^{-1} b,\ p_w = k_1,\ p_x = -\lambda^{-1} a,\ p_y = (k_2 + k_1^2 \lambda\mu AB)/k_1,\ \text{and } p_z = k_1 \mu B,$$

satisfies $K \circ g_k = k$ and $f_k \circ g_k = id$, $g_k \circ f_k = id$. Since g_k is an algebraic map, it is smooth, and so we see $\mathcal{O}_k$ is diffeomorphic to $T^*\mathbf{R}^2$.

The commutation relations for the Poisson bracket on $\mathfrak{t}_4^*$ allow us to compute that $\{f_k^* a, f_k^* A\}_k = \{f_k^* b, f_k^* B\}_k = 1$ and all other Poisson brackets are zero. It follows that $f_k^* : (C^\infty(T^*\mathbf{R}^2), [,]) \to (C^\infty(\mathcal{O}_k), \{,\}_k)$ is a Lie algebra isomorphism. Hence $f_k : \mathcal{O}_k \to T^*\mathbf{R}^2$ is a symplectomorphism. $\square$

2.3. **The hamiltonians.** Let $a_{ij} > 0$ be constants such that $a_{13}a_{34} = a_{12}a_{24}$ and let

$$4H(p) = a_{12}p_x^2 + a_{23}p_y^2 + a_{13}p_z^2 + a_{24}p_u^2 + a_{34}p_v^2 + a_{14}p_w^2 \tag{3}$$

Since the vector field E_H is unaffected by the addition of a Casimir, the term $a_{14}p_w^2$ can be ignored.

Let us introduce a symplectic change of variables on $T^*\mathbf{R}^2$: $A = \frac{1}{\sqrt{2}}(X - Y)$, $B = \frac{1}{\sqrt{2}}(X + Y)$, $a = \frac{1}{\sqrt{2}}(x - y)$, $b = \frac{1}{\sqrt{2}}(x + y)$, $z = c$ and $Z = C$. Because $a_{13}a_{34} = a_{12}a_{24}$, there exists unique $\lambda, \mu > 0$ so that $0 = a_{34}\mu^{-2} - a_{12}\lambda^{-2} = a_{13}\mu^2 - a_{24}\lambda^2$, and $a_{12}\lambda^{-2} + a_{34}\mu^{-2} = 1$. Indeed, we can choose $\lambda^2 = 2a_{12}$ and $\mu^2 = 2a_{34}$. Then:

$$2\mathbf{H}_k = (x^2 - \xi X^2 + \nu X^4) + (y^2 + \omega Y^2 + \nu Y^4 - 2\nu X^2 Y^2), \tag{4}$$

where $\xi = -(a_{13}a_{34}k_1^2 + a_{23}k_2\sqrt{a_{12}a_{34}})$, $\omega = \xi + 2a_{13}a_{34}k_1^2 = a_{13}a_{34}k_1^2 - a_{23}k_2\sqrt{a_{12}a_{34}}$ and $\nu = a_{12}a_{23}a_{34}k_1^2$. Note that we can write $\omega = \xi + 2c\nu$ where $c = \frac{a_{13}}{a_{12}a_{23}}$.

Lemma 2.2. *The Euler vector field of H on the regular coadjoint orbit $\mathcal{O}_k$ (equation 3) is a time change of the hamiltonian vector field of*

$$2\mathbf{H} = x^2 + \left(X^2 - \frac{1}{2}\right)^2 + y^2 + \alpha^2 Y^2 + Y^4 - 2X^2Y^2 \tag{5}$$

on $T^*\mathbf{R}^2$, where $\alpha^2 = 1 + 2c\nu^{\frac{1}{3}}$.

Proof. Define the coordinate change $g_\nu(x,X,y,Y) = (ax, a^{-1}X, ay, a^{-1}Y)$ where $a = \nu^{\frac{1}{6}}$. Then $g^*(\mathbf{H}_k) = a^2\mathbf{H}$. $\square$

Lemma 2.3. *For all $\epsilon > 0$, the hamiltonian flow of $\mathbf{H}$ (equation 5) is conjugate to the flow of the vector field*

$$(6)\quad \mathcal{X}_\epsilon = \left\{ \begin{array}{lllllll} \dot{X} & = & x, & \qquad & \dot{Y} & = & y, \\ \dot{x} & = & X - 2X^3 + 2\epsilon XY^2, & & \dot{y} & = & \left[-\alpha^2 + 2X^2\right] Y + 2\epsilon Y^3. \end{array} \right.$$

Proof. Introduce the coordinate transformation $h_\epsilon(x,X,y,Y) = (x, X, \sqrt{\epsilon}y, \sqrt{\epsilon}Y)$. $\square$

Remark. It is clear from (6) that the vector field $\mathcal{X}_\epsilon$ depends on the parameter α. Inspection of the formula for α (lemma 2.2 and immediately above) shows that α is identically unity when the coefficient a_{13} vanishes. This is the case for the standard subriemannian metric, where $a_{12} = a_{23} = a_{34} = 1$ and the other coefficients vanish. Rather than specializing to $\alpha = 1$, we have elected to carry α through our analysis. The rationale for this will be apparent in section 3.3.

3. Analysis of $\mathcal{X}_\epsilon$

For $\epsilon = 0$, the map h_ϵ is singular. However, the vector field $\mathcal{X}_0$ is well-defined. We will show that $\mathcal{X}_0$ has a normally hyperbolic invariant manifold S whose stable and unstable manifolds coincide, and that this manifold S persists for $\epsilon > 0$ but the stable and unstable manifolds $W^\pm_\epsilon(S)$ no longer coincide. This implies that the Euler vector field $E_H|\mathcal{O}_k$ has transverse homoclinic points for all regular coadjoint orbits.

3.1. **The normally hyperbolic manifold S.**

Inspection of the vector field $\mathcal{X}_\epsilon$ shows that the set

$$S = \{(x,X,y,Y) \ : \ x = X = 0\},$$

is invariant for all ϵ. One sees that for $\epsilon = 0$, the vector field is

$$\mathcal{X}_0 = \left\{ \begin{array}{lllllll} \dot{X} & = & x, & \qquad & \dot{Y} & = & y, \\ \dot{x} & = & X + O(X^3), & & \dot{y} & = & \left[-\alpha^2 + 2X^2\right] Y, \end{array} \right.$$

which shows that S is normally hyperbolic. Therefore S is normally hyperbolic for all ϵ sufficiently small. Since $\mathcal{X}_\epsilon$ is conjugate to the same vector field for all non-zero ϵ, one concludes that S is a normally hyperbolic manifold for *all* ϵ.

3.2. **The stable and unstable manifolds of S.**

The function $h = x^2 + (X^2 - \frac{1}{2})^2$ is a first integral of $\mathcal{X}_0$. The set $h^{-1}(\frac{1}{4})$ is the stable and unstable manifold of S, which we denote by $W^\pm_0(S)$. On $W^\pm_0(S) - S$, the flow of $\mathcal{X}_0$ satisfies

$$(7)\quad \left\{ \begin{array}{lllllll} X & = & \pm\operatorname{sech}(t+t_0), & \qquad & x & = & \mp\tanh(t+t_0)\operatorname{sech}(t+t_0)^2, \\ Y & = & c_0Y_0(t+t_0) + c_1Y_1(t+t_0), & & y & = & \dot{Y}, \end{array} \right.$$

where $X(0) = \pm\operatorname{sech}(t_0), x(0) = \mp\tanh(t_0)\operatorname{sech}(t_0)^2$ and Y_j solves the initial-value problem

$$(8)\quad \left\{ \begin{array}{l} \ddot{Y} + \left[\alpha^2 - 2\operatorname{sech}(t)^2\right] Y = 0, \qquad (*) \\ \\ Y(0) = 1 - j, \quad \dot{Y}(0) = j \end{array} \right.$$

while $Y(0) = c_0Y_0(t_0) + c_1Y_1(t_0)$, $y(0) = c_0\dot{Y}_0(t_0) + c_1\dot{Y}_1(t_0)$. The solutions Y_j are chosen so that they are even ($j = 0$) and odd ($j = 1$) functions of time.

3.3. **The Melnikov function.** To determine if the flow of $\mathcal{X}_\epsilon$ has transverse homoclinic points for non-zero ϵ, we appeal to the following theorem.

Theorem 3.1. *Let $\varphi_\epsilon : M \times \mathbf{R} \to M$ be a complete, smooth flow that depends smoothly on ϵ. Assume that φ_0 possesses a normally hyperbolic, invariant manifold $S \subset M$, and that there is a smooth function $h : M \to \mathbf{R}$ such that*

(1) *the stable and unstable manifolds of S coincide and equal $h^{-1}(0)$;*
(2) *$\mathrm{d}h$ does not vanish on $W_0^\pm(S) - S$.*

Then, for all sufficiently small non-zero ϵ, φ_ϵ possesses a normally hyperbolic invariant manifold S_ϵ and the local stable and unstable manifolds of S_ϵ ($W_\epsilon^+(S)$ and $W_\epsilon^-(S)$, respectively) can be written as the graph of a function $s_\epsilon^\pm : W_0^\pm(S) \to W_\epsilon^\pm(S)$. The splitting distance, defined for $p \in W_0^\pm(S)$ by $s_\epsilon(p) = h \circ s_\epsilon^+(p) - h \circ s_\epsilon^-(p)$, is a smooth function of ϵ and $s_\epsilon(p) = \epsilon m(p) + O(\epsilon^2)$ where

$$m(p) = \int_{t\in\mathbf{R}} \langle \mathrm{d}h, \mathcal{Y}\rangle \circ \varphi_0^t(p)\ \mathrm{d}t, \tag{9}$$

and $\mathcal{Y} = \left.\dfrac{\partial}{\partial\epsilon}\dfrac{\partial}{\partial t}\right|_{t=\epsilon=0} \varphi_\epsilon^t$.

Proof. The proof of this theorem is a standard application of invariant manifold theory plus an adaptation of the proof of the Melnikov formula [7, 11, 4]. □

Remark. If m changes sign, then, for all $\epsilon \neq 0$ sufficiently small, the perturbed stable and unstable manifolds intersect but do not coincide. In our case, a result of Burns and Weiss implies that the topological entropy is non-zero. If m has a non-degenerate zero, then the implicit function theorem implies that, for all $\epsilon \neq 0$ sufficiently small, $W_\epsilon^+(S)$ has a transverse intersection with $W_\epsilon^-(S)$. Note that the intersection of the surface S with a constant energy level is a periodic orbit. Therefore each trajectory in the intersection is doubly asymptotic to a periodic orbit in S.

For the flow defined by $\mathcal{X}_\epsilon$, we have that

$$\mathcal{Y} = \left\{ \begin{array}{lllllll} \dot X &=& 0, & \qquad & \dot Y &=& 0, \\ \dot x &=& 2XY^2, & & \dot y &=& 2Y^3. \end{array}\right. \tag{10}$$

Whence

$$\langle \mathrm{d}h, \mathcal{Y}\rangle = 4xXY^2, \tag{11}$$

since $h = x^2 + (X - \frac{1}{2})^2$. The equations in (7) imply that the Melnikov function is

$$m(p) = c_0c_1 \times \int_{\tau\in\mathbf{R}} -4\ \tanh(\tau)\ \mathrm{sech}(\tau)^2\ Y_0(\tau)\, Y_1(\tau)\ \mathrm{d}\tau. \tag{12}$$

Remarks. (1) The formula for the Melnikov integral (12) appears to be a function on S not $W_0^\pm(S) - S$. This does not contradict Theorem (3.1). Inspection of the integral (equation 9) shows that $m(\varphi_0^s(p)) = m(p)$ for all s and p. The coordinate system on $W_0^\pm(S) - S$ determined by (7) uses time along the flow as one coordinate (t_0), so only the other two coordinates, c_0 and c_1, ought to appear in the Melnikov function. (2) If we write $m(p) = 2c_0c_1 \times I$, where

$$I = \int_0^\infty -4\ \tanh(\tau)\ \mathrm{sech}(\tau)^2\ Y_0(\tau)\, Y_1(\tau)\ \mathrm{d}\tau, \tag{13}$$

then m has non-degenerate zeros along $\{c_0 = 0, c_1 \neq 0 \text{ or } c_1 = 0, c_0 \neq 0\}$, provided that $I \neq 0$.

3.4. **The Legendre functions and I.** Substitution of $z = \tanh(t)$ transforms the differential equation (8*) into the Legendre differential equation

$$(1-z^2)Y'' - 2zY' + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right) Y = 0, \tag{14}$$

where $\mu = i\alpha$, $\nu = -\frac{1}{2} + \frac{\sqrt{-7}}{2}$ and $' = \frac{\mathrm{d}}{\mathrm{d}z}$. The integral I (equation 13) is transformed to

$$I = \int_0^1 z\, U_0(z)\, U_1(z)\, \mathrm{d}z, \tag{15}$$

where $U_j(z) = Y_j(t)$.

3.5. **The Melnikov function is non-zero: $I \neq 0$.** For the remainder of this note, $q : [0,\infty) \to \mathbf{R}$ is a continuously differentiable function such that $\lim_{t\to\infty} q(t) = 0$ and $\alpha > 0$ is a fixed positive number. The function $z = z(t)$ is assumed to solve

$$\ddot{z} + [\alpha^2 - q(t)]\, z = 0. \tag{16}$$

In analogy with the integral I in equation (13), define an integral

$$I = \int_0^\infty \dot{q}(t)\, z_0(t)\, z_1(t)\, \mathrm{d}t \tag{17}$$

I is implicitly a function of α; one wants to prove that I can vanish at most countably many times.

Let us first prove the following.

Lemma 3.2. *If z solves equation (16), then z is bounded with bounded derivative.*

Proof. Define $H = \frac{1}{2}(\alpha^2 z^2 + \dot{z}^2)$. From (16) one computes that $\dot{H} = qz\dot{z}$. Integrating by parts yields $H = C_0 + \frac{1}{2}q(t)z(t)^2 + \int_0^t -\frac{1}{2}\dot{q}(s)\, z(s)^2\, \mathrm{d}s$, where C_0 is a constant that depends only on $z(0)$ and $\dot{z}(0)$. Thus

$$\frac{1}{2}\left[\alpha^2 - q(t)\right]\, z(t)^2 \leq C_0 + \int_0^t -\frac{1}{2}\,\dot{q}(s)\, z(s)^2\, \mathrm{d}s. \tag{18}$$

Since $q(t) \to 0$ as $t \to \infty$, there is a $T \geq 0$ such that $\alpha^2 - q(t) \geq \frac{1}{2}\alpha^2$ for all $t \geq T$. Therefore, equation (18) implies that there is a constant C_1 such that for all $t \geq 0$

$$z(t)^2 \leq C_1 + \frac{4}{\alpha^2} \times \int_0^t -\dot{q}(s)\, z(s)^2\, \mathrm{d}s \tag{19}$$

This is a Gronwall inequality for $u = z^2$. Thus

$$z(t)^2 \leq C_1 \exp(-\frac{4}{\alpha^2} q(t))$$

for all $t \geq 0$. Since q is continuous and converges to 0 at infinity, it is bounded. Therefore, z is bounded.

To prove that $w = \dot{z}$ is bounded, define $H = \frac{1}{2}(\alpha^2 w^2 + \dot{w}^2)$. Since $\ddot{w} + [\alpha^2 - q(t)]w = f$, $f = \dot{q}z$, one computes that $\dot{H} = qw\dot{w} + f\dot{w}$. One can bound $\int_0^t f(s)\dot{w}(s)\mathrm{d}s$ using that $\ddot{z} = \dot{w}$ and that q and z are already bounded. One then obtains a Gronwall inequality like (19) for $w(t)^2$. $\square$

Lemma 3.3. *If z_0, z_1 are solutions to equation (16), then the limit*

$$I = -\lim_{t\to\infty} \left[\dot{z}_0(t)\, \dot{z}_1(t) + \alpha^2 z_0(t)\, z_1(t)\right] \tag{20}$$

exists and equals $W\alpha\cot(B)$ where the angle B is defined below, and W is the Wronskian of the solutions z_0, z_1.

Proof. Let z be any solution of (16). Let t_n be the sequence of zeros of $z(t)$, indexed in increasing order. Let $\phi_n = \alpha t_n \bmod 2\pi$ and $a_n = \dot{z}(t_n)$. There is a sequence n_k such that $\phi_{n_k} \to \phi \bmod 2\pi$ and $a_{n_k} \to a > 0$. The former follows by compactness of $\mathbf{R}/2\pi\mathbf{Z}$ and the latter because $\dot{z}$ is bounded.

Since $q(t) \to 0$, the Sturm comparison theorem implies that the sequence ϕ_n converges to ϕ and $t_{n+1} - t_n$ converges to π/α [13]. One sees that positive and negative zeros of z must alternate for all n. Hence, without loss of generality, one may assume that $\dot{z}(t_{2n}) \to a$ and $\dot{z}(t_{2n+1}) \to -a$. The continuous dependence of solutions on initial data therefore implies that $z^n(t) := z(t + \frac{2n\pi}{\alpha})$ converges in the weak Whitney C^1 topology to $a\sin(\alpha t - \phi)$. In particular, $z^n(t)$ converges uniformly to $a\sin(\alpha t - \phi)$ for $t \in [0, 4\pi/\alpha]$.

To apply these observations to the limit (20), let N be sufficiently large so that the n-th and $n+1$-th zeros of both z_0 and z_1 are at most $\frac{2\pi}{\alpha}$ apart for all $n \geq N$. Let $s \in [\frac{2n\pi}{\alpha}, \frac{2(n+1)\pi}{\alpha}]$ and write $s = t + \frac{n\pi}{\alpha}$ so that $t \in [0, \frac{2\pi}{\alpha}]$. Then

$$\begin{aligned}
&|\dot{z}_0(s)\dot{z}_1(s) + \alpha^2 z_0(s) z_1(s) - \alpha^2 a_0 a_1 \cos(\phi_1 - \phi_0)| \\
&\leq |\dot{z}_0^n(t)\dot{z}_1^n(t) - \alpha^2 a_0 a_1 \cos(\alpha t - \phi_0)\cos(\alpha t - \phi_1)| + \\
&\quad \alpha^2 |z_0^n(t) z_1^n(t) - a_0 a_1 \sin(\alpha t - \phi_0)\sin(\alpha t - \phi_1)|.
\end{aligned}$$

If $s \to \infty$, then $n \to \infty$. The above-mentioned uniform convergence for $t \in [0, 2\pi/\alpha]$ shows that the limit (20) exists and equals $A\cos(B)$ where $A = \alpha^2 a_0 a_1$ and $B = \phi_1 - \phi_0 \bmod 2\pi$.

On the other hand, the Wronskian W of z_0, z_1 is constant and

$$W = z_0(t)\dot{z}_1(t) - z_1(t)\dot{z}_0(t) \xrightarrow{t\to\infty} \alpha a_0 a_1 \sin(\phi_1 - \phi_0),$$

by the same argument as above. Therefore $I/W = \alpha\cot(B)$. □

Remarks. (1) The angle B has the following interpretation which emerges from the proof of lemma (3.3). The zeros of solutions to (16) are asymptotically π/α apart, and the zeros of linearly independent solutions are interlaced. The angle B is defined so that $B/\alpha \bmod \pi/\alpha$ is asymptotically the time between consecutive zeros of the linearly independent solutions. Figure 1, left, plots B as a function of α for the solutions $z_j = Y_j$ to the initial-value problem (8). One expects that as $\alpha \to \infty$, the solutions should converge quite quickly to cos and sin, whence B should approach $\frac{\pi}{2}$. The figure captures this behaviour quite nicely. (2) The function $I = W\alpha\cot(B)$ from lemma 3.3 can be computed numerically. The Sturm comparison theorem implies that the n-th zero t_n of a solution z satisfies $\pi/\alpha < t_{n+1} - t_n < \pi/\alpha \times (1 + q/\alpha^2)$ if $|q(t)| < \alpha^2/2$ for all $t > t_n$. If q goes to zero sufficiently fast, one can numerically compute the first several zeros and obtain a reasonably accurate estimate of B. Figure 1, right, shows the graph of I for $q = 2\,\mathrm{sech}(t)^2$.

Lemma 3.4. *If z_0, z_1 are solutions to (16) such that $\dot{z}_0$ and z_1 vanish at $t = 0$, then the integral*

$$I = \int_0^\infty \dot{q}(t)\, z_0(t)\, z_1(t)\, \mathrm{d}t \tag{17}$$

exists and equals $W\alpha\cot(B)$ where the angle B is described in Lemma 3.3, and W is the Wronskian of the solutions z_0, z_1.

Proof. By lemma 3.2, both solutions are bounded, so one can apply integration by parts to the integral. This yields

$$\begin{aligned} I &= q(0)z_0(0)z_1(0) - \int_0^\infty q(t)\ [\dot{z}_0(t)\, z_1(t)\ + z_0(t)\, \dot{z}_1(t)]\ \mathrm{d}t, \\ &= -\int_0^\infty q(t)\ [\dot{z}_0(t)\, z_1(t)\ + z_0(t)\, \dot{z}_1(t)]\ \mathrm{d}t \end{aligned}$$

since z_1 vanishes at $t = 0$.

From (16), it is known that $q(t)z_0(t) = \ddot{z}_0(t) + \alpha^2 z_0(t)$ and similarly for z_1. Therefore

$$\begin{aligned} I &= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\ \left[\dot{z}_0(t)\, \dot{z}_1(t)\ + \alpha^2 z_0(t)\, z_1(t)\right]\ \mathrm{d}t, \\ &= -\lim_{t\to\infty} \left[\dot{z}_0(t)\, \dot{z}_1(t)\ + \alpha^2 z_0(t)\, z_1(t)\right], \qquad \text{since } \dot{z}_0(0) = 0 = z_1(0), \\ &= W\alpha \cot(B) \qquad\qquad \text{by lemma (3.3).} \end{aligned}$$

□

Lemma 3.5. *Assume that there exists $C, \lambda > 0$ such that $|q(t)| < Ce^{\lambda t}$ for all $t > 0$. Then the integral $I = I(\alpha)$ is a holomorphic function of α on the strip $|\mathrm{Im}\,\alpha| < \lambda$ about the real line.*

Consequently, if q is an even, monotone function, then I vanishes countably many times at most.

Proof. A solution $z = z(t;\alpha)$ to (16) is a holomorphic function of α for each fixed t [13]. For large t and $|\mathrm{Im}\,\alpha| < \lambda$, the solution $z = z(t;\alpha)$ is equal to $\cos(\alpha t + \phi)$ plus a term that grows slower than $e^{\lambda t}$. This implies, by the residue formula, that $I = I(\alpha)$ is holomorphic provided that $|\mathrm{Im}\,\alpha| < \lambda$.

When $\alpha = 0$ and q is even, the even and odd solutions to (16) do not change sign. Therefore, if q is monotone, then the integrand defining $I(0)$ does not change sign, so $I(0) \neq 0$. Thus, I can vanish at most countably many times on the strip $|\mathrm{Im}\,\alpha| < \lambda$. □

Theorem 1.1. If $a_{13} \neq 0$ – whence $c \neq 0$ in Lemma 2.2 –, then lemma 3.5 shows that the hamiltonian flow of H (equation 3) on all but countably many coadjoint orbits in $\mathfrak{t}_4^*$ has a horseshoe. This proves the main result, Theorem 1.1. □

4. The degenerate case when $\alpha \equiv 1$

If $a_{13} = 0$, as occurs for the Carnot subriemannian metric of [10], then $\alpha \equiv 1$ and lemma 3.5 cannot be applied. We investigate two distinct ways to address this problem. The first is direct and numerical; the second leads to some further insight into the integral I.

4.1. **Numerical evidence.** In this case, figure 1 indicates that $I(1)$ is approximately -2.75. Table 1 shows the results of a numerical computation of $I(1)$ with varying step sizes. It is clear from this table that $I(1) = -2.76$ to two decimal places.

To estimate the error in the computations, one uses the fact that the differential equations (16) are hamiltonian with the hamiltonian

$$\mathsf{H} = \frac{1}{2}p^2 + \frac{1}{2}\left[\alpha^2 - q(\tau)\right]\ z^2 + u, \tag{21}$$

where p, z and u, τ are canonically conjugate variables (along solutions, $\tau = \tau_0 + t$, so it is a pseudo-time). Since H is preserved along solutions to (16), the maximum deviation of H along a numerical solution provides an estimate of the upper bound of the error in the solutions z_0, z_1.

4.2. **Qualitative evidence.** As explained in the Remark in subsection 3.5, one may compute I as a function of α by computing the phase angle B. Figure 1 graphs B and I versus α. This figure shows that $I(1)$ does not vanish.

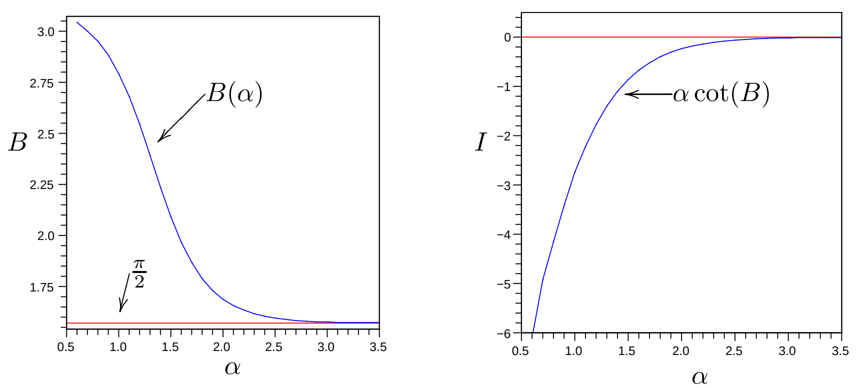

FIGURE 1. Left: Phase angle B vs. α; Right: $I = \alpha\cot(B)$ vs. α. Both plots use the solutions $z_j = Y_j$ of (8) with $q(t) = 2\,\mathrm{sech}(t)^2$. These solutions are computed numerically in Maple using the 4-th order Runge-Kutta method; the zeros are located by interval halving.

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TABLE 1. The numerical calculation of I with $\alpha = 1$.

h	I	$\mathsf{H}_{0,min}$	$\mathsf{H}_{0,max}$	$\mathsf{H}_{1,min}$	$\mathsf{H}_{1,max}$	$\mathsf{H}_{0,max} - \mathsf{H}_{0,min}$	$\mathsf{H}_{1,max} - \mathsf{H}_{1,min}$
0.5	-2.76812630	-0.5	-0.49025150	0.49833857	0.51067514	0.00974849	0.01233657
0.25	-2.76366763	-0.5	-0.49944022	0.49992738	0.50063022	0.00055977	0.00070283
0.125	-2.76340793	-0.5	-0.49996571	0.49999582	0.50003878	$3.42847126 \times 10^{-5}$	$4.29572646 \times 10^{-5}$
0.0625	-2.76339200	-0.5	-0.49999786	0.49999974	0.50000241	$2.13207876 \times 10^{-6}$	$2.67223559 \times 10^{-6}$
0.03125	-2.76339101	-0.5	-0.49999986	0.49999998	0.50000015	$1.33088170 \times 10^{-7}$	$1.66835298 \times 10^{-7}$
0.015625	-2.76339095	-0.5	-0.49999999	0.49999999	0.50000000	$8.31541421 \times 10^{-9}$	$1.04238692 \times 10^{-8}$
0.0078125	-2.76339094	-0.5	-0.49999999	0.49999999	0.50000000	$5.19672801 \times 10^{-10}$	$6.51456639 \times 10^{-10}$

Solutions to the hamiltonian equations of H are computed with the Forest-Ruth 4-th order symplectic integrator [12] and initial conditions $z(0) = j, \dot{z}(0) = 1 - j, \tau(0) = 0, u(0) = 0$ for $j = 0, 1$. The maximum (resp. minimum) value of H along the j-th numerical solution over the interval $[0, 35]$ is indicated by $\mathsf{H}_{j,max}$ (resp. $\mathsf{H}_{j,min}$). The integral I is computed by

$$I^h = h \times \sum_{i=0}^{N} \dot{q}(t_i)\, z_0^h(t_i)\, z_1^h(t_i)$$

where z_j^h is the computed solution with step size h, $N = 35/h$, $t_i = i \times h$ and $q(t) = 2\,\mathrm{sech}(t)^2$. Files at `http://www.maths.ed.ac.uk/~lbutler/t4.html`